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AN INTEGRAL EQUATION FOR A PROBLEM OF THE INDENTATION OF A WEDGE-SHAPED PUNCH†

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An integral equation, to which one can reduce the problem of the indentation of a wedge-shaped punch into an elastic half-space when there is no friction, is considered. Using a multidimensional analogue of the Wiener–Hopf factorization method, an explicit formula for solving an equation close to that under consideration (a perturbed equation) is obtained.

A method of solving the corresponding integral equation for a wedge-shaped punch was proposed in [1] (see also [2, 3]). It was based on expansion of the kernel and the use of the Mellin transform. Another method is employed below to consider the same integral equation.

1. FORMULATION OF THE PROBLEM

In the case of static contact between a wedge-shaped punch and an elastic half-space without friction [1, p. 118] the problem can be reduced to solving the following integral equation

$$\iint_{\Omega} k(x-\xi, y-\eta)q(\xi,\eta)d\xi d\eta = C_1 f(x,y), \quad (x,y)\in\Omega \tag{1.1}$$

where $f(x, y)$ is a given function in the two-dimensional domain $\Omega = \{(x,y) \in \mathbf{R}^2: y > a|x|, a > 0\}$, $q(\xi, \eta)$ is an unknown function, C_1 is determined by the constants of elasticity of the half-space, and the kernel of the integral equation (1.1) is defined by the inverse Fourier transform

$$k(x,y) = \frac{1}{4\pi^2}\iint_{\mathbf{R}^2} (\xi^2+\eta^2)^{-1/2} e^{-i(x\xi+y\eta)} d\xi d\eta$$

Here, as compared to the equation in [1], some slight changes in the definition of Ω are made for convenience. For the domain under consideration the wedge-shaped punch must be convex, which was not required in [1]. However, it will be shown below that the problem can be solved by a similar method in the case of a non-convex wedge.

In general, Eq. (1.1) belongs to the class of pseudodifferential equations [4]. It can of course be considered from the viewpoint of integral equations, but to do so one must pay special attention to the meaning of the integral in the equation.

2. FUNCTIONAL SPACES AND PSEUDODIFFERENTIAL OPERATORS

Equation (1.1) will be considered within the framework of the theory of pseudodifferential equations in the Sobolev–Slobodetskii spaces H^s. Let us briefly recall the basic definitions.

The Sobolev–Slobodetskii space $H^s(\mathbf{R}^2) \equiv H^s$, $s \in \mathbf{R}$ consists of distributions $u(x, y)$ whose Fourier transform F is a locally Lebesgue integrable function $\tilde{u}(\xi, \eta)$ such that

$$\|u\|_s^2 = \iint_{\mathbf{R}^2} |\tilde{u}(\xi,\eta)|^2 (1+|\xi|+|\eta|)^{2s} d\xi d\eta < +\infty$$

We recall that if $S(\mathbf{R}^2)$ is the Schwartz class of infinitely differentiable functions decreasing along with all their derivatives more rapidly than any negative power of $|x| + |y|$ as $|x| + |y| \to +\infty$, then the Fourier transform of $u(x, y) \in S(\mathbf{R}^2)$ is defined by

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$$\tilde{u}(\xi,\eta) \equiv (Fu)(\xi,\eta) = \iint_{\mathbf{R}^2} e^{i(x\xi+y\eta)}u(x,y)dxdy$$

Finally, we note that if $b \in S'(\mathbf{R}^2)$ is a distribution, then a pseudodifferential operator A can be defined by

$$Au = F^{-1}(b\tilde{u}), \quad u \in S(\mathbf{R}^2) \tag{2.1}$$

The function $b(\xi, \eta)$ is called the symbol of A.

We introduce the following operator generated by the kernel $k(x, y)$ in (1.1)

$$(Ku)(x,y) = \iint_{\mathbf{R}^2} k(x-x', y-y')u(x',y')dx'dy' \tag{2.2}$$

According to (2.1), K can be regarded as a pseudodifferential operator with symbol $(\xi^2 + \eta^2)^{-1/2}$ and (2.2) as the integral representation of K.

3. PERTURBATION AND FACTORIZATION

Consider the operator K_ε corresponding to the symbol

$$\sigma(\xi,\eta,\varepsilon) = (\xi^2+\eta^2+\varepsilon^2)^{-1/2}, \quad \varepsilon \in \mathbf{R}, \quad \varepsilon \neq 0$$

We will write Eq. (1.1) in operator form

$$P_+Kq = f, \quad (P_+u)(x,y) = \begin{cases} u(x,y), & (x,y) \in \Omega \\ 0, & (x,y) \notin \overline{\Omega} \end{cases} \tag{3.1}$$

where P_+ is the restriction operator to Ω, and replace it by the perturbed equation

$$P_+K_\varepsilon p_+ = f \tag{3.2}$$

(the constant C_1 is omitted in (3.1) and (3.2); the subscript plus is attached to p to indicate that p is defined in Ω). The change of equation is justified by the fact that K is an unbounded operator in H^s, while the operators K_ε have the required property, namely, they are bounded operators from H^s to H^{s+1} [4, Lemma 4.4, p. 45].

Equation (3.2) will be solved exactly for all $\varepsilon \neq 0$. If $p_+^{(\varepsilon)}$ is a solution of Eq. (3.2) and $\lim p_+^{(\varepsilon)} = q$ as $\varepsilon \to 0$ exists in the H^s-norm, one can verify that q satisfies Eq. (3.1). Because of this $p_+^{(\varepsilon)}$ will be called an approximate solution of Eq. (3.1) (even though, in general, the exact solution does not exist a priori for an arbitrary right-hand side $f \in H^{s+1}(\Omega)$; here and henceforth $H^s(\Omega)$ will denote the space of functions $u \in H^s$ such that $\operatorname{supp} u \subset \Omega$).

We will represent $\sigma(\xi, \eta, \varepsilon)$ in the form

$$\begin{aligned} &\sigma(\xi,\eta,\varepsilon) = \sigma^+(\xi,\eta,\varepsilon)\sigma^-(\xi,\eta,\varepsilon) \\ &\sigma^\pm(\xi,\eta,\varepsilon) = (\sqrt{a^2+1}\,\eta \pm \zeta(\xi,\eta,\varepsilon))^{-1/2}, \quad \zeta(\xi,\eta,\varepsilon) = \sqrt{a^2\eta^2-\xi^2-\varepsilon^2} \end{aligned} \tag{3.3}$$

The representation (3.3) will be called the wave factorization of $\sigma(\xi, \eta, \varepsilon)$. The factors σ^+ and σ^- have a number of remarkable properties. Let O^+ be a cone in $\mathbf{R}^2$ of the form $\{(\xi, \eta): a\eta > |\xi|\}$ and let O^- be the opposite cone $\{(\xi, \eta): (-\xi, \eta) \in O^+\}$. Let $T(O^\pm)$ be the radial tubular domain over $O^\pm$ [5], that is, the set $\mathbf{R}^2 + iO^\pm$ in $\mathbf{C}^2$. It turns out that $\sigma^\pm(\zeta, \eta, \varepsilon)$ is the boundary value of a function analytic in $T(O^\pm)$ and having no zeros in $T(O^\pm)$. Indeed, let us consider the function

$$F(z_1,z_2) = (\sqrt{a^2+1}\,z_2 + \zeta(z_1,z_2,\varepsilon))^{-1/2} \tag{3.4}$$

of two complex variables z_1 and z_2. We set $T(O) = T(O^+) \cup T(O^-)$. We know that the function $a^2z_2^2 - z_1^2$ has no non-negative values for $z = (z_1, z_2) \in T(O)$ (in [5], the lemma on p. 350, this is proved for a light cone with $a = 1$; the assertions of the lemma can easily be extended to the case of an arbitrary $a > 0$. The same must obviously be true for $a^2z_2^2 - z_1^2 - \varepsilon^2$. It follows that each of these two branches of $\zeta(z_1, z_2, \varepsilon)$ is single-valued and analytic in $T(O)$. Moreover

$$|\zeta(z_1, z_2, \varepsilon)| \leqslant d(1 + |z|)^{\alpha} \tag{3.5}$$

where d is a positive constant depending only on a and where ε, $\alpha = 1$. This implies that $\zeta(z_1, z_2, \varepsilon)$ has boundary values for $(\xi', \eta') \to 0$, $(\xi', \eta') \in O^+$, $((\xi', \eta') \to 0, (\xi', \eta') \in O^-)$ in the sense of distributions [5, p. 275], $z \in \mathbf{C}^2$ being represented by $(\xi, i\xi', \eta + i\eta')$.

The boundary values are independent of the way (ξ', η') tends to zero. It is therefore convenient to consider limits of the form $\lim_{\eta' \to 0}\zeta(\xi, \eta + i\eta', \varepsilon)$.

Choosing one of the branches of $\zeta(z_1, z_2, \varepsilon)$, for example, that which maps $T(O)$ onto the upper half-plane, it can be shown that

$$\zeta(\xi, \eta \pm i0, \varepsilon) = \begin{cases} \pm\sqrt{\kappa}, & \kappa > 0, \quad \eta > 0 \\ \mp\sqrt{\kappa}, & \kappa > 0, \quad \eta < 0 \\ i\sqrt{-\kappa}, & \kappa < 0, \quad (\kappa = a^2\eta^2 - \xi^2 - \varepsilon^2) \end{cases}$$

will be the boundary values of this branch. Note that

$$\zeta(\xi, \eta \pm i0, \varepsilon) = -\overline{\zeta(\xi, \eta - i0, \varepsilon)}$$

(the bar denotes complex conjugation).

Let us return to the function (3.4). It is analytic in $T(O^+)$, since we have $\operatorname{Im} z_2 > 0$ and $\operatorname{Im} \zeta(z_1, z_2, \varepsilon) > 0$ for $z \in T(O^+)$, i.e. all the values of $\sqrt{(a^2 + 1)}z_2 + \zeta(z_1, z_2, \varepsilon)$ lie in the upper half-plane. In the same way we can verify that $(\sqrt{(a^2 + 1)}z_2 + \overline{\zeta(z_1, z_2, \varepsilon)})^{-1/2}$ is analytic in $T(O^-)$. It follows that if $\zeta(\xi, \eta + i0, \varepsilon)$ is taken as $\zeta(\xi, \eta, \varepsilon)$, then the factors σ^+ and σ^- in (3.3) have the properties which will be needed in what follows, namely, $(\sigma^+)^{\pm 1}$ and $(\sigma^-)^{\pm 1}$ admit of analytic continuation into $T(O^+)$ and $T(O^-)$, respectively, and the analytic continuations satisfy (3.5) for some $\alpha \geqslant 0$ and have no zeros in $T(O^+)$, $T(O^-)$ (one can take $\alpha = 1/2$ for $(\sigma^+)^{-1}$, $(\sigma^-)^{-1}$ and $\alpha = 0$ for σ^+ and σ^-).

4. INVESTIGATION OF EQUATION (3.2)

We shall seek a solution of Eq. (3.2) in the space $H^s(\Omega)$ of distributions u. The right-hand side f in (3.2) will be considered in the space $H_1^{s+1}(\Omega)$ of functions from $S'(\Omega)$ admitting of an extension to $H^{s+1}(\mathbf{R}^2)$ with

$$\|f\|_{H_1^{s+1}(\Omega)} = \inf\|lf\|_{H^{s+1}(\mathbf{R}^2)}$$

by definition, the infimum being taken over all possible continuations l.

Let f_1 be an arbitrary continuation of f from Ω to $\mathbf{R}^2$, $f_1 \in H^{s+1}(\mathbf{R}^2)$. We set $p_- = f_1 - K_\varepsilon p_+$ and rewrite Eq. (3.2) in the form

$$K_\varepsilon p_+ + p_- = f_1 \tag{4.1}$$

On applying a Fourier transformation to Eq. (4.1), we can write

$$\sigma(\xi, \eta, \varepsilon)\tilde{p}_+(\xi, \eta) + \tilde{p}_-(\xi, \eta) = \tilde{f}_1(\xi, \eta) \tag{4.2}$$

Taking the wave factorization (3.3) into account, the last equation can be represented as

$$\tilde{q}_+(\xi, \eta) + \tilde{q}_-(\xi, \eta) = \tilde{f}_2(\xi, \eta) \tag{4.3}$$

$$\left(\tilde{q}_+(\xi, \eta) = \sigma^+(\xi, \eta, \varepsilon)\tilde{p}_+(\xi, \eta), \quad \tilde{q}_-(\xi, \eta) = \frac{\tilde{p}_-(\xi, \eta)}{\sigma^-(\xi, \eta, \varepsilon)}, \quad \tilde{f}_2(\xi, \eta) = \frac{\tilde{f}_1(\xi, \eta)}{\sigma^-(\xi, \eta, \varepsilon)}\right)$$

(Here the dependence of $\tilde{q}_+$ and $\tilde{q}_-$ on ε is omitted for simplicity.)

Equation (4.3) is the multidimensional analogue of the so-called jump problem [6]. The one-dimensional version of it arises naturally in the well-known Wiener–Hopf method [1, 7, 8].

Let us analyse Eq. (4.3) in detail. We denote by $\widetilde{H}_+^s$ the Fourier image of $H^s(\Omega)$, by $\widetilde{H}_-^s$ the Fourier image of $H^s(\mathbf{R}^2 \setminus \overline{\Omega})$, and by $\widetilde{H}^s$ the Fourier image of H^s. Our immediate goal is to verify that $\tilde{q}_+ \in \widetilde{H}_+^{s+1/2}$ and $\tilde{q}_- \in \widetilde{H}_-^{s+1/2}$. We note right now that $\tilde{q}_+, \tilde{q}_-, \tilde{f}_2 \in \widetilde{H}^{s+1/2}$, since $p_+ \in H^s$ and the pseudo-differential operator with symbol σ^+ has order $-1/2$ [4, Lemma 4.4, p. 45], $f_1, p_- \in H^{s+1}$, and the pseudo-differential operator with symbol $(\sigma^-)^{-1}$ has order $1/2$.

Furthermore, $\widetilde{H}_+^{s+1/2}$ can be described explicitly [9, Section 10]: it consists of the boundary values (in the sense of distributions) of analytic functions $\tilde{u}(\xi, \eta)$ in $T(O^+)$ with finite norm

$$\sup\left(\iint_{\mathbf{R}^2} |\tilde{u}(\xi + i\xi', \eta + i\eta')|^2 \lambda^{2s+1} d\xi d\eta\right)^{1/2}, \quad (\xi', \eta') \in O^+$$

$$\lambda = 1 + |\xi| + |\eta|$$

which is identical with

$$\left(\iint_{\mathbf{R}^2} |\tilde{u}(\xi, \eta)|^2 \lambda^{2s+1} d\xi d\eta\right)^{1/2}$$

Since $\tilde{p}_+$ and σ^+ are the boundary values of analytic functions on $T(O^+)$ and

$$\iint_{\mathbf{R}^2} |\sigma^+(\xi, \eta, \varepsilon)|^2 |\tilde{p}_+(\xi, \eta)|^2 \lambda^{2s+1} d\xi d\eta \leq d \iint_{\mathbf{R}^2} |\tilde{p}_+(\xi, \eta)|^2 \lambda^{2s} d\xi d\eta < +\infty$$

(because $\tilde{p}_+ \in \widetilde{H}_+^s$), it follows that $\tilde{q}_+ \in \widetilde{H}_+^{s+1/2}$.

We consider $\tilde{q}_-$. Since $(\sigma^-)^{-1}(\xi, \eta, \varepsilon)$ admits of an analytic continuation into $T(O^-)$ and satisfies (3.5) with $\alpha = 1/2$, the inverse Fourier transform $(\sigma^-)^{-1}(\xi, \eta, \varepsilon)$ in the sense of distributions is a distribution Λ concentrated on $\overline{\Omega}_1 = \{(x, y) \in \mathbf{R}^2 \colon y \leq -a|x|\}$ and $(\sigma^-)^{-1}\tilde{p}_-$ is the convolution of Λ and p_-. Approximating p_- by infinitely differentiable functions compactly supported on $\mathbf{R}^2 \setminus \Omega$ and taking into account that $\operatorname{supp} \Lambda \subset \overline{\Omega}_1$ and $\operatorname{supp} p_- \subset \mathbf{R}^2 \setminus \Omega$, one can verify that $\operatorname{supp} \Lambda * p_- \subset \mathbf{R}^2 \setminus \Omega$ (the asterisk denotes convolution). The latter is equivalent to the fact that $(\sigma^-)^{-1}\tilde{p}_- \in \widetilde{H}^{s+1/2}$.

Equation (4.3) will be solved using an integral operator, which will now be introduced and whose basic properties will be described.

5. THE INTEGRAL OPERATOR

We define the integral operator

$$(G_2 u)(\xi, \eta) = \lim_{\tau \to 0+} \iint_{\mathbf{R}^2} \frac{u(x, y) dx dy}{(\xi - x)^2 - a^2(\eta - y + i\tau)^2}$$

on functions from the Schwartz class $S(\mathbf{R}^2)$. If $\theta(x, y)$ is the characteristic function of Ω, it can be shown that

$$F(\theta u) = 2aG_2\tilde{u}, \quad \forall u \in S(\mathbf{R}^2) \tag{5.1}$$

Indeed, consider the integral ($\tau > 0$)

$$\iint_{\mathbf{R}^2} e^{i(x\xi + y\eta)} \theta(x, y) u(x, y) e^{-\tau y} dx dy$$

This is the Fourier transform of the product of two functions $u(x, y)$ and $\theta(x, y)e^{-\tau y}$ which are absolutely integrable (the latter enables us to apply the theorem on convolution).

We find the Fourier transform of $\theta(x, y)e^{-\tau y}$

$$\iint_{\mathbf{R}^2} e^{i(x\xi + y\eta)} \theta(x, y) e^{-\tau y} dx dy = \iint_{\mathbf{R}^2} e^{ix\xi} e^{iy(\eta + i\tau)} dx dy =$$

$$= \int_{-\infty}^{+\infty} \left(\int_{a|x|}^{+\infty} e^{iy(\eta + i\tau)} dy\right) e^{ix\xi} dx = -\frac{1}{i(\eta + i\tau)} \int_{-\infty}^{+\infty} e^{ia|x|(\eta + i\tau)} e^{ix\xi} dx = \frac{2a}{\xi^2 - a^2(\eta + i\tau)^2}$$

Convoluting the latter function with $\tilde{u}(\xi, \eta)$ and taking the limit as $\tau \to 0$ ($\tau > 0$) we obtain (5.1).

The operator G_2 can be written in a more customary form if the substitutions

$$\xi' = \xi - a\eta, \quad \eta' = \xi + a\eta, \quad x' = x - ay, \quad y' = x + ay \tag{5.2}$$

are made for the variables. Then

$$(G_2 u)\left(\frac{\eta'+\zeta'}{2}, \frac{\eta'-\xi'}{2a}\right) = \frac{1}{2a}\lim_{\tau\to 0+}\iint_{\mathbf{R}^2}\frac{u_1(x',y')dx'dy'}{(\xi'-x'-i\tau)(\eta'-y'+i\tau)}$$

$$u_1(x,y) = u\left(\frac{y+x}{2}, \frac{y-x}{2a}\right)$$

The right-hand side of this equality is the product of two one-dimensional Cauchy integrals, and the limit is easy to compute. We introduce the operators

$$(S_1 u)(x',y') = \text{v.p.}\frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{u(\xi',y')d\xi'}{x'-\xi'}$$

$$(S_2 u)(x',y') = \text{v.p.}\frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{u(x',\eta')d\eta'}{y'-\eta'}, \quad S_{12} = S_1 S_2$$

We have [6, p. 83]

$$(G_2 u)\left(\frac{\eta'+\zeta'}{2}, \frac{\eta'-\xi'}{2a}\right) = \\ = -\frac{\pi^2}{2a}(-u_1(\xi',\eta') + (S_1 u_1)(\xi',\eta') - (S_2 u_1)(\xi',\eta') + (S_{12}u_1)(\xi',\eta')) \tag{5.3}$$

The representation (5.3) is convenient because it easily enables us to reduce the problem concerned with the boundedness of G_2 in the $\tilde{H}^s$ norm for $|s| < 1/2$ to the corresponding one-dimensional results [4, p. 59], since $1 + |x'| + |y'| \sim 1 + |x| + |y|$, $1 + |\xi'| + |\eta'| \sim 1 + |\xi| + |\eta|$ with the substitution (5.2), i.e. the ratio of the numbers in question has upper and lower bounds of positive constants.

Indeed, the boundedness of G_2 in the norm of $\tilde{H}^s$ is equivalent to the boundedness of

$$(G_2^0 u)(\xi,\eta) = \lim_{\tau\to 0+}\iint_{\mathbf{R}^2}\frac{u(x,y)(1+|x|+|y|)^s\,dxdy}{(1+|\xi|+|\eta|)^s((\xi-x)^2 - a^2(\eta-y+i\tau)^2)}$$

in the space $L_2(\mathbf{R}^2)$ of functions with finite norm

$$\|u\|_{L_2(\mathbf{R}^2)} = \left(\iint_{\mathbf{R}^2}|u(x,y)|^2\,dxdy\right)^{1/2}$$

Since, according to (5.3), G_2 is a linear combination of S_1 and S_2 and their products (apart from a change of variables leading to an equivalent H^s norm), it remains to use the appropriate result from [4] (Theorem 5.1).

Finally, the last important property is that G_2 is an orthogonal projection from $\tilde{H}^s$ onto $\tilde{H}^s_+$ (it should be mentioned once more that $|s| < 1/2$). If θ is the operator of multiplication by $\theta(x, y)$, then one can verify that it is bounded in the H^s norm. This follows from (5.1), the boundedness of G_2 in the $\tilde{H}^s$ norm and the fact that $S(\mathbf{R}^2)$ is dense in H^s. For $u(x, y) \in S(\mathbf{R}^2)$ we have supp $\theta(x, y)u(x, y) \subset \bar{\Omega}$. It follows that $\theta u \in H^s(\Omega)$. That $H^s(\Omega)$ is closed in H^s can be verified by a method analogous to that in [4]. Thus G_2 is a bounded operator from $\tilde{H}^s$ to $\tilde{H}^s_+$. In precisely the same way one can verify that $I - G_2$, where I is the identity operator, is a bounded operator from $\tilde{H}^s$ to $\tilde{H}^s_-$.

This property implies the same fact which will be used as the basis for solving Eq. (4.3). Namely, for

$|s| < 1/2$ every function $\tilde{u}(\xi, \eta) \in \tilde{H}^s$ can be uniquely represented in the form

$$\tilde{u}(\xi,\eta) = \tilde{u}_1(\xi,\eta) + \tilde{u}_2(\xi,\eta), \quad \tilde{u}_1 \in \tilde{H}^s_+, \ \tilde{u}_2 \in \tilde{H}^s_-$$

with $\tilde{u}_1 = G_2\tilde{u}$ and $\tilde{u}_2 = (I - G_2)\tilde{u}$.

The equality

$$u = \theta u + (1 - \theta)u \tag{5.4}$$

is obvious. Changing to Fourier transforms taking (5.1) into account and extending (5.4) by continuity from $S(\mathbf{R}^2)$ to H^s, we can write

$$\tilde{u} = G_2\tilde{u} + (I - G_2)\tilde{u} \tag{5.5}$$

and set $G_2\tilde{u} = \tilde{u}_1$ and $(I - G_2)\tilde{u} = \tilde{u}_2$.

It remains to verify that (5.5) is a unique representation. To do this it suffices to show that if

$$\tilde{u}_1 + \tilde{u}_2 = 0 \tag{5.6}$$

then $\tilde{u}_1 \equiv 0$ and $\tilde{u}_2 \equiv 0$.

Taking $\tilde{g}_1 \in C_0^\infty(\Omega)$ and $\tilde{g}_2 \in C_0^\infty(\mathbf{R}^2 \backslash \bar{\Omega})$ (by the methods of [4, p. 44] it can be shown that the classes $C_0^\infty(\Omega)$ and $C_0^\infty(\mathbf{R}^2 \backslash \bar{\Omega})$ of compactly supported infinitely differentiable functions whose support is contained in the given domain are dense, respectively, in H^s_+, H^s_-), we have

$$\theta g_1 = g_1, \quad \theta g_2 = 0$$

Approximately u_1 by functions of type g_1 and u_2 by functions of type g_2 and using the properties of G_2, we obtain

$$G_2\tilde{u}_1 = \tilde{u}_1, \quad G_2\tilde{u}_2 = 0 \tag{5.7}$$

By analogy

$$(I - G_2)\tilde{u}_1 = 0, \quad (I - G_2)\tilde{u}_2 = \tilde{u}_2 \tag{5.8}$$

Applying (5.7) and (5.8) to (5.6), we obtain the required result.

6. SOLUTION OF EQUATION (3.2)

According to Section 5, the solution of Eq. (4.5) for $|s + 1/2| < 1/2$, i.e. $-1 < s < 0$, has the form

$$\tilde{q}_+(\xi,\eta) = (G_2\tilde{f}_2)(\xi,\eta)$$

It follows that the Fourier transform of the solution of Eq. (3.2) has the explicit form

$$\tilde{p}_+(\xi,\eta) = (\sigma^+)^{-1}(\xi,\eta,\varepsilon) \lim_{\tau\to 0+} \iint_{\mathbf{R}^2} \frac{\tilde{f}_1(x,y)dxdy}{\sigma^-(x,y,\varepsilon)((\xi - x)^2 - a^2(\eta - y + i\tau)^2)} \tag{6.1}$$

where f_1 is an arbitrary continuation of f from Ω to $\mathbf{R}^2$ with $f_1 \in H^{s+1}$, p_+ being independent of the choice of this continuation.

We write (6.1) in the symbolic form

$$\tilde{p}_+ = (\sigma^+)^{-1} G_2 (\sigma^-)^{-1} \tilde{f}_1 \tag{6.2}$$

Let f'_1 be another continuation of f

$$\tilde{p}'_+ = (\sigma^+)^{-1} G_2 (\sigma^-)^{-1} \tilde{f}'_1 \tag{6.3}$$

Subtracting (6.2) from (6.3), we get

$$\tilde{p}'_+ - \tilde{p}_+ = (\sigma^+)^{-1} G_2 (\sigma^-)^{-1} (\tilde{f}_1 - \tilde{f}_1)$$

But supp $(f'_1 - f_1) \subset \mathbf{R}^2 \backslash \Omega$ and then $(\sigma^-)^{-1}(f'_1 - \tilde{f}_1) \in \tilde{H}^s_-$ (see Section 4), which implies (Section 5) that $G_2(\sigma^-)^{-1}(\tilde{f}'_1 - \tilde{f}_1) = 0$. This means that $\tilde{p}'_+ = \tilde{p}_+$.

By the boundedness of G_1 in $\tilde{H}^s$ for $|s| < 1/2$ and the boundedness of the pseudodifferential operator of order $1/2$ with symbols $(\sigma^+)^{-1}$ and $(\sigma^-)^{-1}$ acting from H^s to $H^{s-1/2}$, the a priori estimate

$$\|p_+\|_{H^s(\Omega)} \leq d\|lf\|_{H^{s+1}} \leq d'\|f\|_{H_1^{s+1}(\Omega)}$$

of the solution follows from (6.2). The rightmost inequality holds because (6.2) is independent of the choice of the continuation lf, and so lf can be chosen in such a way that the desired inequality is satisfied.

We remark that if Ω is a non-convex cone (the angle at the vertex being greater than π), the method presented is obviously applicable without any major modifications: G_2 should be introduced as it applies to the cone $\mathbf{R}^2 \backslash \overline{\Omega}$, $I - G_2$ should be used in Section 6, and the wave factorization of σ should be constructed for the cones related to $\mathbf{R}^2 \backslash \overline{\Omega}$.

The case $a = 0$ when Ω "degenerates" into the half-plane, it would seem, is not covered by the proposed scheme. It is, in fact, much simpler and can be considered using the standard Wiener–Hopf method, in which factorization is performed relative to one variable, the other one playing the role of a parameter (see [4]). The function $\sigma(\xi, \eta, \varepsilon)$ should be represented in the form

$$\sigma(\xi, \eta, \varepsilon) = (\eta + i\sqrt{\xi^2 + \varepsilon^2})^{-1/2} (\eta - i\sqrt{\xi^2 + \varepsilon^2})^{-1/2}$$

so that the factors admit of analytic continuation into the upper and lower half-planes with respect to η with ξ fixed, satisfy the required estimates, and have no zeros.

Remarks. 1. It is easy to take the limit as $\varepsilon \to 0$ in (6.1), namely, one should set $\varepsilon = 0$ in (6.1). The point is that all the operators on the right-hand side of (6.1) are bounded in the corresponding spaces H^s for $\varepsilon = 0$, and so (6.1) makes sense for $\varepsilon = 0$. The perturbation of the original equation is necessary in order that the operators encountered at each intermediate stage should be bounded.

2. The problem concerned with the asymptotic behaviour (near the sides) of the solution calls for a special study. Nevertheless, the root singularity encountered in contact problems is indeed present in the case when $a = 0$ (see [4, p. 93, Theorem 9.1 et seq.]; in this case one should take the index κ appearing there to be $1/2$, i.e. the degree of σ^+).

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